New Darboux Transformation for Hirota-Satsuma coupled KdV System

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Abstract

A new Darboux transformation is presented for the Hirota-Satsuma coupled KdV system. It is shown that this Darboux transformation can be constructed by means of two methods: Painlevé analysis and reduction of a binary Darboux transformation. By iteration of the Darboux transformation, the Grammian type solutions are found for the coupled KdV system.

1 Introduction

Hirota and Satsuma [1] proposed the very first coupled KdV system, which describes interactions of two long waves with different dispersion relations. They then constructed the three-soliton solutions and five conserved quantities for this system. In a following paper [2], these authors shown that the coupled KdV system is the four-reduction of the celebrated KP hierarchy and its soliton solutions can be derived from those of the KP equation.

It is known that a solitonic equation normally possesses a Lax pair. Using the Wahlquist-Estabrook prolongation procedure, Dodd and Fordy [3] found a Lax representation for the Hirota-Satsuma coupled KdV (HS-KdV) system. Meanwhile, Wilson [4] observed that the HS-KdV system is just an example of many integrable systems arose from the Drinfeld-Sokolov theory. A Bäcklund transformation is constructed for this system by Levi [5]. Very recently, some new solutions for HS-KdV system are presented [6].

Most integrable equations have Darboux Transformations (DT), which are very effective to construct solutions. We notice that Leble and Ustinov [7] found a DT for the HS-KdV system. They started with an elementary DT for a more general spectral problem, then they found that a proper reduction led to a DT for HS-KdV system. In general, an integrable system may possess more than one DT. For example, the KdV equation has classical DT and binary DT as well. We are interested in finding a new DT for the HS-KdV system. We will show that this new DT appears in two ways, from Painlevé analysis and from reduction of the binary DT for a more general spectral problem (we refer to [8]-[10] and the references there for Painlevé analysis approach to DT).

The paper is set out as follows. In section 2 we construct a new DT for the HS-KdV system within the framework of Painlevé analysis. In section 3, we show that a DT can be obtained from the reduction of a general DT, then we prove that this DT is equivalent to the one found in section 2. In section 4 we discuss the iteration of the DT and show that the solutions for the HS-KdV system can be represented as Grammians. Final section presents our conclusion.

2 DT from Painlevé analysis

The HS-KdV system reads

$$u_t = \frac{1}{2}u_{xxx} + 3uu_x - 6vv_x, \quad v_t = -v_{xxx} - 3uv_x. \tag{1}$$

where the subscripts denote partial derivatives.

A detailed Painlevé analysis of this system was performed by Weiss [11][12]. He found that this system has the Painlevé property. Indeed, there are two branches and one of them, principal branch, is

$$u = \tau^{-2}\sum_{j=0}^{\infty} u_j\tau^j, \quad v = \tau^{-1}\sum_{j=0}^{\infty} v_j\tau^j, \tag{2}$$

and the resonances are

$$j = -1, 0, 1, 4, 5, 6$$

By truncating the expansions (2) on the constant level, one has a transformation

$$u = u_2 + 2(\ln \tau)_{xx}, \quad v = v_1 + \frac{v_0}{\tau} \tag{3}$$

where u_2 and v_1 solve the HS-KdV system (1) and τ, u_2, v_0, v_1 have to satisfy the following overdetermined system

$$\begin{aligned}
&\tau_t + \tau_{xxx} + 3\tau_x u_2 = 2\tau_x \vartheta, \\
&v_0 = \tau_x H, \\
&\frac{\tau_t}{\tau_x} - \frac{1}{2}\{\tau; x\} = \frac{3}{4}H^2 + \vartheta, \\
&\vartheta_x^2 = (\lambda^2 + \vartheta^2)H^2, \\
&v_1 = -\frac{v_{0x}}{2\tau_x} - \frac{1}{3}(\lambda^2 + \vartheta^2)^{\frac{1}{2}}, \\
&H_t = -\left[H_{xx} + \frac{1}{4}H^3 + \vartheta H + \frac{2}{3}\{\tau; x\}H\right]_x,
\end{aligned}$$

where $\{\tau; x\} = \left(\frac{\tau_{xx}}{\tau_x}\right)_x - \frac{1}{2}\left(\frac{\tau_{xx}}{\tau_x}\right)^2$ is the Schwarzian derivative.

To obtain a Lax pair for the HS-KdV system (1), Weiss introduced a new variable

$$W = \frac{\tau_{xx}}{\tau_x} \tag{4}$$

and derived the equations for H and W, which are modified Hirota-Satsuma system. The Miura map between HS-KdV system and its modification is

$$\begin{aligned}
u_2 &= -\frac{1}{2}\left[W_x + \frac{1}{2}W^2 + \frac{1}{2}H^2 - \frac{2}{3}\vartheta\right], \\
v_1 &= -\frac{1}{2}\left[H_x + WH + \frac{2}{3}(\lambda^2 + \vartheta^2)^{1/2}\right].
\end{aligned}$$

Taking

$$W + H = 2\frac{\eta_{1x}}{\eta_1}, \quad W - H = 2\frac{\eta_{2x}}{\eta_2}, \tag{5}$$

$$\vartheta = \lambda \sinh \alpha, \quad \alpha = \ln(\frac{\eta_1}{\eta_2}), \tag{6}$$

one arrives at the linear problem

$$\eta_{1xx} + (u_2 + v_1)\eta_1 = -\lambda \eta_2, \tag{7}$$

$$\eta_{2xx} + (u_2 - v_1)\eta_2 = \lambda \eta_1, \tag{8}$$

and

$$\eta_{1t} = -\frac{1}{2}(u_{2x} - 2v_{1x})\eta_1 + (u_2 - 2v_1)\eta_{1x} - 2\lambda \eta_{2x}, \tag{9}$$

$$\eta_{2t} = -\frac{1}{2}(u_{2x} + 2v_{1x})\eta_2 + (u_2 + 2v_1)\eta_{2x} + 2\lambda \eta_{1x}, \tag{10}$$

where the parameter λ is scaled for convenience. This is the linear problem found by Dodd and Fordy via the prolongation approach [3].

We intend to construct a DT for the HS-KdV systems (1). As a first step, we rewrite the transformation (3) between two solutions of the system (1) in terms of the solutions of linear problem (7-10). Using (4) and (5), we obtain

$$u - u_2 = 2(\ln \tau)_{xx}, \quad v - v_1 = \frac{\eta_2 \eta_{1x} - \eta_{2x}\eta_1}{\tau}, \tag{11}$$

where (η_1, η_2) is a solution of the linear system (7-10). The x-derivative of function τ is defined in terms of η_1 and η_2 as follows

$$\tau_x = \eta_1 \eta_2. \tag{12}$$

The transformation (11) is the Darboux transformation on the potential level. Next, we need to find the transformations for the wave function. Let us suppose that (ϕ_1, ϕ_2) solves the linear systems (7-10) with $\lambda = \mu$. For the new wave function, we make the following Ansätz

$$\hat{\phi}_1 = \phi_1 + \frac{f}{\tau}, \quad \hat{\phi}_2 = \phi_2 + \frac{g}{\tau}, \tag{13}$$

where $f = f(x,t)$ and $g = g(x,t)$ are the functions to be determined. By requiring the new wave function $(\hat{\phi}_1, \hat{\phi}_2)$ solves the linear problem (7-10) with replacement $u_2 \to u$, $v_1 \to v$ and $\lambda \to \mu$, we are led to two equations

$$\begin{aligned} &\phi_{1xx} + (u_2 + v_1)\phi_1 + \mu\phi_2 + \\ &\quad + \frac{1}{\tau}\Big[f_{xx} + (u_2 + v_1)f + 2\tau_{xx}\phi_1 + (\eta_{1x}\eta_2 - \eta_1\eta_{2x})\phi_1 + \mu g\Big] + \\ &\quad\quad + \frac{1}{\tau^2}\Big[-2f_x\tau_x + f\tau_{xx} - 2\tau_x^2\phi_1 + (\eta_{1x}\eta_2 - \eta_1\eta_{2x})f\Big] = 0, \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\phi_{2xx} + (u_2 - v_1)\phi_2 - \mu\phi_1 + \\ &\quad + \frac{1}{\tau}\Big[g_{xx} + (u_2 - v_1)g + 2\tau_{xx}\phi_2 - (\eta_{1x}\eta_2 - \eta_1\eta_{2x})\phi_2 - \mu f\Big] + \\ &\quad\quad + \frac{1}{\tau^2}\Big[-2g_x\tau_x + g\tau_{xx} - 2\tau_x^2\phi_2 - (\eta_{1x}\eta_2 - \eta_1\eta_{2x})g\Big] = 0. \end{aligned} \tag{15}$$

Let us consider the equation (14) first. The first part of this equation, the coefficient of τ^0, is identically zero. Equating the coefficients of τ^{-1} and τ^{-2} to zero respectively, we have

$$f_{xx} + (u_2 + v_1)f + 2\tau_{xx}\phi_1 + (\eta_1\eta_2 - \eta_1\eta_{2x})\phi_1 + \mu g = 0, \tag{16}$$

$$-2f_x\tau_x + f\tau_{xx} - 2\tau_x^2\phi_1 + (\eta_{1x}\eta_2 - \eta_1\eta_{2x})f = 0. \tag{17}$$

From (17), we obtain

$$f_x - f\frac{\eta_{1x}}{\eta_1} + \eta_1\eta_2\phi_1 = 0. \tag{18}$$

Now, using (16) and (18), we have

$$\lambda\eta_2 f - \mu\eta_1 g = \eta_1\eta_2(\phi_1\eta_{1x} - \phi_{1x}\eta_1). \tag{19}$$

A similar consideration for the other equation (15) provides us

$$-\mu\eta_2 f+\lambda\eta_1 g=\eta_1\eta_2(\eta_2\phi_{2x}-\eta_{2x}\phi_2). \tag{20}$$

In this way, we obtain a system of linear equations for f and g, namely, (19) and (20). Solving this linear system, we get the expressions for f and g as follows

$$f=\frac{\eta_1}{\lambda^2-\mu^2}\Big[\mu(\eta_2\phi_{2x}-\eta_{2x}\phi_2)+\lambda(\phi_1\eta_{1x}-\phi_{1x}\eta_1)\Big], \tag{21}$$

$$g=\frac{\eta_2}{\lambda^2-\mu^2}\Big[\mu(\phi_1\eta_{1x}-\phi_{1x}\eta_1)+\lambda(\eta_2\phi_{2x}-\eta_{2x}\phi_2)\Big]. \tag{22}$$

Thus, we obtain the explicit expressions for the transformations for the wave functions, that is, the equations (13) with f and g given by (21-22). However, this is not the end of story: we have to make sure that the transformed wave functions and fields also satisfy the temporal part of the linear problem. But first we need to obtain the time evolution of τ, which is

$$\tau_t=2\lambda(\eta_1^2-\eta_2^2)-2\eta_{1x}\eta_{2x}-u_2\eta_1\eta_2. \tag{23}$$

Verifying that the new wave functions and new fields obtained under the transformation are indeed the solutions for transformed linear systems is too tedious to perform by pen and paper, but we do check its validity by means of MAPLE. Thus, we obtain a new DT for the HS-KdV system (1). We summarize our results as

Proposition *Let (η_1,η_2) be the solutions of the linear systems (7-10) and (ϕ_1,ϕ_2) be the solutions of the same linear systems with the spectral parameter μ. Let the function τ be defined by equations (12) and (23). Then the new field variables and the new wave functions defined via (11) and (13) with f and g given by (21) and (22) solve the linear system (7-10) with μ as the spectral parameter.*

3 DT from reduction

Our DT constructed above is different from the known DT obtained by Leble and Ustinov [7]. Indeed, their DT is of kind classical DT while our one is of kind binary DT: we used both the wave functions and the adjoint wave functions in our construction. Furthermore, our approach is based on Painlevé analysis while they obtain their DT by means of reductions. Noticing these differences, we are interested in constructing a DT for (1) by reducing the binary DT for a general linear problem.

As in [7], we consider the following linear problem

$$(\partial^2+F\partial+U)\Psi=\mu\sigma_3\Psi, \tag{24}$$

where μ is the spectral parameter and

$$F=\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix},\quad U=\begin{pmatrix} u_{11} & u_{12}\\ u_{21} & u_{22}\end{pmatrix},\quad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{25}$$

If

$$F=0,\quad u_{11}=u_{22}=u,\quad u_{12}=u_{21}=v, \tag{26}$$

above linear problem is equivalent to the spatial part of the linear problem for the HS-KdV system given in last section.

The binary Darboux transformation has been constructed in general case, for example, [13] contains a detailed discussion on this issue. In the present case, we consider

$$L\Theta \equiv \Big[\partial_y - \sigma_3(\partial^2 + F\partial + U)\Big]\Theta = 0, \tag{27}$$

where y is a new independent variable, F and U are those matrices given by (25). It is easy to see that (24) is a dimension reduction of (27).

What we will do next is to study reduction of the general binary DT and construct a proper DT for HS-KdV system (1). To construct a binary DT. we also need to consider the adjoint linear problem

$$\Big[-\partial_y - (\partial^2 - \partial F^t + U^t)\sigma_3\Big]\Phi = 0. \tag{28}$$

Now we take Θ as a matrix solution of the linear system (27) and Φ as a matrix solution of the adjoint linear system (28). Then we define a potential matrix by

$$\Omega_x = \Phi^t\Theta,$$

here and in the sequel superscript ^{t} denotes matrix transposition.

It is known that

$$G = 1 - \Theta\Omega^{-1}\partial^{-1}\Phi^t$$

satisfies the following equation

$$(\partial_y - \sigma_3(\partial^2 + \hat{F}\partial + \hat{U}))G = G(\partial_y - \sigma_3(\partial^2 + F\partial + U)),$$

where the hatted matrices $\hat{F}$ and $\hat{U}$ are the 2×2 matrices with the new transformed variables. Therefore, one has a binary DT (for wave functions)

$$\hat{\Psi} = G\Psi$$

in this general case. As argued in [13], this general DT preserves certain properties of the operator. Therefore, we choose in the sequel the adjoint wave function as

$$\Phi = -A\Theta, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This reduction is consistent with the DT. Furthermore, a simple calculation shows that under this reduction F is not alerted

$$\hat{F} - F = \Theta\Omega^{-1}\Phi^t - \sigma_3\Theta\Omega^{-1}\Phi^t\sigma_3 = 0,$$

and the potential matrix Ω is found to satisfy

$$\Omega_x = \det(\Theta)A. \tag{29}$$

We now have a DT for the linear problem (27) with $F = 0$. Since the HS-KdV system (1) is a $1+1$-dimensional system, we have to make further reduction. That is, we need to do dimensional reduction. Furthermore, U has to be symmetric and its diagonal entries have to be identical, namely, U need to take the form (26). To this end, we take

$$\Theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix},$$

where (θ_1, θ_2) is the solution of the following system with $\mu = \lambda$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{xx} + \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \mu\sigma_3 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$
$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}u_x & v_x \\ v_x & -\frac{1}{2}u_x \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \begin{pmatrix} u+2\mu & -2v \\ -2v & u-2\mu \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x.$$

In this case to show that the general DT is reducible, one has to prove the following identity holds

$$\sigma_3(\partial^2 + \hat{U})(1 - \Theta\Omega^{-1}\partial^{-1}\Theta A) = (1 - \Theta\Omega^{-1}\partial^{-1}\Theta A)\sigma_3(\partial^2 + U),$$

where $\hat{U} = \begin{pmatrix} \hat{u} & \hat{v} \\ \hat{v} & \hat{u} \end{pmatrix}$. Above equation yields

$$\hat{U} - U = (\Theta\Omega^{-1}\Theta A)_x + \sigma_3(\Theta\Omega^{-1}\Theta A)_x\sigma_3 + (\Theta\Omega^{-1})_x\Theta A - \sigma_3(\Theta\Omega^{-1})_x\Theta A\sigma_3, \tag{30}$$

and

$$\begin{aligned} &\sigma_3(\Theta_{xx} + U\Theta)\Omega^{-1}\partial^{-1}\Theta A - \Theta\Omega^{-1}\partial^{-1}(\Theta_{xx}A\sigma_3 + \Theta A\sigma_3 U) + \\ &+\Big[\Theta\Omega^{-1}\Theta_x A\sigma_3\Theta - \sigma_3\Theta\Omega^{-1}\Theta A\Theta_x\Big]\Omega^{-1}\partial^{-1}\Theta A = 0. \end{aligned} \tag{31}$$

The validity of above identity (31) can be checked by straightforward calculation. Thus, we have a DT for the linear problems (24):

$$\hat{\Psi} = (1 - \Theta\Omega^{-1}\partial^{-1}\Theta A)\Psi, \tag{32}$$

and the transformation for U is given by (30). This shows that the general Darboux transformation can be reduced to the special case: the HS-KdV case. Of course, we also have to work out the time evolution part of the DT. This can be easily done by means of MAPLE.

To compare this DT with the DT obtained in last section, we rewrite above results in terms of scalars rather than vectors. From the transformation for U (30), we readily have

$$\hat{u} - u = 2(\ln\rho)_{xx}, \quad \hat{v} - v = 2\frac{\theta_1\theta_{2x} - \theta_{1x}\theta_2}{\rho},$$

where

$$\rho_x = \theta_1^2 - \theta_2^2,$$

is derived from (29). The transformation (32) for the wave functions is

$$\hat{\phi}_1 = \phi_1 + \frac{p}{\rho}, \quad \hat{\phi}_2 = \phi_2 + \frac{q}{\rho},$$

where

$$\begin{aligned} p &= \frac{\theta_2}{\lambda+\mu}(\theta_2\phi_{1x} - \theta_{2x}\phi_1 + \theta_1\phi_{2x} - \theta_{1x}\phi_2) + \frac{\theta_1}{\lambda-\mu}(\theta_1\phi_{1x} - \theta_{1x}\phi_1 + \theta_2\phi_{2x} - \theta_{2x}\phi_2), \\ q &= \frac{\theta_1}{\lambda+\mu}(\theta_2\phi_{1x} - \theta_{2x}\phi_1 + \theta_1\phi_{2x} - \theta_{1x}\phi_2) + \frac{\theta_2}{\lambda-\mu}(\theta_1\phi_{1x} - \theta_{1x}\phi_1 + \theta_2\phi_{2x} - \theta_{2x}\phi_2). \end{aligned}$$

For the temporal part, we need time evolution for ρ, which is

$$\rho_t = 2\lambda(\theta_1^2+\theta_2^2)+\frac{1}{2}u(\theta_1^2-\theta_2^2)+\theta_{1x}^2-\theta_{2x}^2.$$

With all these in hand, we can show that the transformed variables and wave functions solve the $t-$part of the linear problem.

Therefore, we do find a DT for the HS-KdV system. The DT presented in this section seems different from the one obtained in last section, but indeed they are same. To establish the equivalence, we need to do the following transformations

$$\theta_1+\theta_2=f_1, \quad \theta_1-\theta_2=f_2, \quad u_2=u, \quad v_1=v,$$

then an easy calculation shows that the DT obtained by reduction is nothing but the one constructed by Painlevé analysis.

4 Iterated DT and Exact Solutions

In general, a DT can be iterated and a compact representation of solutions can be derived in terms of some special determinants such as Wronskian, Grammian. We will show that it is also the case for our DT.

We notice that our DT for the wave functions (13) can be written as

$$\tilde{\phi}_1=\phi_1-\frac{\eta_1}{\tau}\int^x\phi_1\eta_2dx, \qquad \tilde{\phi}_2=\phi_2-\frac{\eta_2}{\tau}\int^x\eta_1\phi_2dx.$$

To do iteration, we take N solutions of the linear system (7-10) $(\eta_1^{[k]},\eta_2^{[k]})$ with respect to λ_k $(k=1,\ldots,N)$. After iterations, we have the new wave functions

$$\phi_1[N] \;=\; \frac{1}{\tau[N]}\begin{vmatrix} \int^x\eta_1^{[1]}\eta_2^{[1]}dx & \cdots & \int^x\eta_1^{[1]}\eta_2^{[N]}dx & \eta_1^{[1]} \\ \vdots & \vdots & \vdots & \vdots \\ \int^x\eta_1^{[N]}\eta_2^{[1]}dx & \cdots & \int^x\eta_1^{[N]}\eta_2^{[N]}dx & \eta_1^{[N]} \\ \int^x\phi_1\eta_2^{[1]}dx & \cdots & \int^x\phi_1\eta_2^{[N]}dx & \phi_1 \end{vmatrix}, \tag{33}$$

$$\phi_2[N] \;=\; \frac{1}{\tau[N]}\begin{vmatrix} \int^x\eta_1^{[1]}\eta_2^{[1]}dx & \cdots & \int^x\eta_1^{[1]}\eta_2^{[N]}dx & \int^x\eta_1^{[1]}\phi_2dx \\ \vdots & \vdots & \vdots & \vdots \\ \int^x\eta_1^{[N]}\eta_2^{[1]}dx & \cdots & \int^x\eta_1^{[N]}\eta_2^{[N]}dx & \int^x\eta_1^{[N]}\phi_2dx \\ \eta_2^{[1]} & \cdots & \eta_2^{[N]} & \phi_2 \end{vmatrix}, \tag{34}$$

and

$$u[N]=u+2(\ln\tau[N])_{xx}$$

where

$$\tau[N]=\det\left(\int^x\eta_1^{[k]}\eta_2^{[\ell]}\right).$$

Although after a single DT, the field variable v has a very nice representation, we do not have a similar one after iterations. However, if we put $u=w_x$, we may solve the first equation of (1) and obtain

$$v^2=\frac{1}{6}(w_{xxx}+3w_x^2-2w_t).$$

In this way, we can find the transformation for the field v. Therefore, the solutions for (1) are represented in terms of a Grammian $\tau[N]$.

As a final part of this section, we generate solutions for the HS-KdV system (1) by means of our DT. Let consider the simplest case: $u_2 = 0$, $v_1 = c$ (c is an arbitrary constant). Thus we need to solve

$$\begin{aligned} \eta_{1xx} &= -c\eta_1 - i\lambda\eta_2, & \eta_{2xx} &= c\eta_2 + i\lambda\eta_1, \\ \eta_{1t} &= -2c\eta_{1x} - 2i\lambda\eta_{2x}, & \eta_{2t} &= 2c\eta_{2x} + 2i\lambda\eta_{1x}, \end{aligned}$$

where for convenience we made a scaling for the spectral parameter: $\lambda \to i\lambda$ ($i^2 = -1$). The general solutions of the above system read as

$$\eta_1 = c_1 e^{\sqrt{\kappa}(2\kappa t + x)} + c_2 e^{-\sqrt{\kappa}(2\kappa t + x)} + c_3 e^{-i\sqrt{\kappa}(-2\kappa t + x)} + c_4 e^{i\sqrt{\kappa}(-2\kappa t + x)}, \tag{35}$$

$$\begin{aligned} \eta_2 = {} & c_1(c+\kappa)e^{\sqrt{\kappa}(2\kappa t + x)} + c_2(c+\kappa)e^{-\sqrt{\kappa}(2\kappa t + x)} \\ & + c_3(c-\kappa)e^{-i\sqrt{\kappa}(-2\kappa t + x)} + c_4(c-\kappa)e^{i\sqrt{\kappa}(-2\kappa t + x)} \end{aligned} \tag{36}$$

where $\kappa^2 = c^2 + \lambda^2$, c_j ($j = 1, \ldots, 4$) are arbitrary constants. Solving

$$\tau_x = \eta_1\eta_2, \quad \tau_t = 2i\lambda(\eta_1^2 - \eta_2^2) - 2\eta_{1x}\eta_{2x},$$

we find the potential

$$\begin{aligned} \tau = {} & \frac{i}{\sqrt{\kappa^2 - c^2}} \Bigg[12\kappa \left[c_1c_2(c+\kappa) + c_3c_4(\kappa - c) \right] t + 2\left[c_1c_2(c+\kappa) - c_3c_4(\kappa - c) \right] x \\ & + \frac{c_1^2(c+\kappa)}{2\sqrt{\kappa}} e^{2\sqrt{\kappa}(2\kappa t + x)} - \frac{c_2^2(c+\kappa)}{2\sqrt{\kappa}} e^{-2\sqrt{\kappa}(2\kappa t + x)} + \frac{ic_3^2(c-\kappa)}{2\sqrt{\kappa}} e^{-2i\sqrt{\kappa}(-2\kappa t + x)} \\ & - \frac{ic_4^2(c-\kappa)}{2\sqrt{\kappa}} e^{2i\sqrt{\kappa}(-2\kappa t + x)} - \frac{(1+i)c_2c_4c}{\sqrt{\kappa}} e^{-\sqrt{\kappa}[2(1+i)\kappa t + (1-i)x]} \\ & - \frac{(1-i)c_2c_3c}{\sqrt{\kappa}} e^{-\sqrt{\kappa}[2(1-i)\kappa t + (1+i)x]} + \frac{(1+i)c_1c_3c}{\sqrt{\kappa}} e^{\sqrt{\kappa}[2(1+i)\kappa t + (1-i)x]} \\ & + \frac{(1-i)c_1c_4c}{\sqrt{\kappa}} e^{\sqrt{\kappa}[2(1-i)\kappa t + (1+i)x]} \Bigg] \end{aligned} \tag{37}$$

and the solutions for (1) are

$$u = (\ln \tau)_{xx}, \qquad v = c + \frac{\eta_{1x}\eta_2 - \eta_1\eta_{2x}}{\tau},$$

where η_1, η_2 and τ are given by (35), (36) and (37) respectively. Our solutions contain the known 1-soliton solutions as special cases. Indeed, by setting $c_1 = c_3 = 0$, we arrive at

$$\begin{aligned} \tau &= -\frac{ic_2^2}{2}\sqrt{\frac{(c+\kappa)}{\kappa(\kappa - c)}} e^{-2\sqrt{\kappa}(2\kappa t + x)} \left[1 + \frac{2(1+i)cc_4}{c_2(c+\kappa)} e^{\theta} + \frac{ic_4^2(c-\kappa)}{c_2^2(c+\kappa)} e^{2\theta} \right], \\ \eta_{1x}\eta_2 - \eta_1\eta_{2x} &= \frac{2(i-1)\kappa^{\frac{3}{2}} c_2c_4}{\sqrt{\kappa^2 - c^2}} e^{-2\sqrt{\kappa}(2\kappa t + x)} e^{\theta} \end{aligned}$$

where $\theta = \sqrt{\kappa}[2(1-i)\kappa t + (1+i)x]$. if we further put $c = 0$, we obtain the 1-soliton solution first found by Hirota and Satsuma [1]. We remark that our DT allows us to construct multi-soliton solutions for HS-KdV system (1).

5 Conclusions

In this paper, we constructed a new DT for the HS-KdV system. We notice that our DT is a binary DT while the one constructed by Leble and Ustinov is a classical DT. The situation is similar to the celebrated KdV case. We obtained our DT by using two methods, namely Painlevé analysis and reduction method. In general, one may obtain a Bäcklund transformation if the wave functions involved in a DT are eliminated. Thus, one may ask if it is possible to obtain a Bäcklund transformation from our new DT. We realize that it is not trivial to do this sort of elimination in the present case. On the other hand, we remark that DT is easier to handle than Bäcklund transformation in order to construct explicit solutions.

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